

# Student project: Of spinning coins and merging black holes

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## Abstract

For the past decade, the SAIL labs at the University of Sydney have been challenging students with short research projects that elucidate basic principles of physics. These include the development of instruments launched on cubesats, balloons, on telescopes or placed out in the field. This experiment is inspired by the spectacular 2015 discovery of merging black holes with the Laser Interferometric Gravitational Observatory (LIGO). Students are profoundly inspired by LIGO, and for good reason, but it is challenging to construct a table top demonstration of a gravitational observatory. Instead we consider chirps which are remarkable transient phenomena in nature involving both frequency and amplitude modulation, as we can demonstrate with a spinning coin. In the case of the LIGO event, orbital energy is being released as gravitational radiation; for the spinning coin, its spin/orbit energy is being released dissipatively (sound, heat, air viscosity). Our experiment involves a simple device to spin a coin remotely. This aids repeatability and allows us to spin the coin within a vacuum chamber to examine the contribution of air viscosity.<sup>1</sup>

## 1 Introduction

Many forms of time-varying oscillatory behaviour give rise to chirps, including speech and music, bat sonar and bird song. Our machines exploit chirp in modern radar systems and in fibre-based telecommunications. Chirps also occur ahead of earthquakes, medical seizures and market crashes. Intriguingly, chirps occur in mathematics, in particular, in relation to critical phenomena and power-law divergences, but also in Weierstrass and Riemann functions.

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<sup>1</sup>Experiment movies are posted at <https://www.youtube.com/channel/UCgEqb638Dp2SHvdwTnsaApA>

The mathematics of chirp is subtle because there are many forms (e.g. linear, power-law, hyperbolic) and time-averaged quantities are less useful than for common band-limited functions (Flandrin 2001). Even so, they have powerful applications in signal decomposition and speech simulation, *inter alia*.

## 2 Chirp signal from the LIGO discovery event

A good student can quickly derive and plot the LIGO chirp for merging black holes using classical physics with help from Mathur et al (2016) and Abbott et al (2016b). To a small factor, much of what is observed can be understood in terms of Newtonian physics as long as one compels the binary system to “radiate” its inertia through the quadrupole moment. A useful analogy is with dipole radiation from a radio beacon due to the changing electric or magnetic dipole moment of the antenna. A fundamental difference is that the frequency of gravitational radiation is twice the frequency of rotation since the quadrupole moment of equal or unequal masses is invariant under a rotation of  $\pi$ .

If two black holes  $M$  and  $m$  are separated by  $R+r$  such that  $Mr = mR$ , Kepler’s laws lead us to

$$\omega^2 = \frac{G(M+m)}{(R+r)^3} \quad (1)$$

for circular orbits; eccentric orbits quickly circularize in any event. The total energy of the system (orbital+gravitational) is

$$E_{\text{tot}} = \frac{1}{2} \frac{GMm}{(R+r)} \quad (2)$$

$$= \frac{1}{2} \frac{G^{2/3} \omega^{2/3} Mm}{(M+m)^{1/3}}. \quad (3)$$

The moment of inertia of the system about the centre of mass is

$$I = \frac{Mm}{M+m} (r+R)^2 \quad (4)$$

which is essentially the mass quadrupole of the binary system. Since the power radiated must be proportional to the square of the amplitude, simple dimensional analysis (Mathur et al 2016) leads to

$$P_{\text{rad}} \propto \frac{GI^2 \omega^6}{c^5} \quad (5)$$

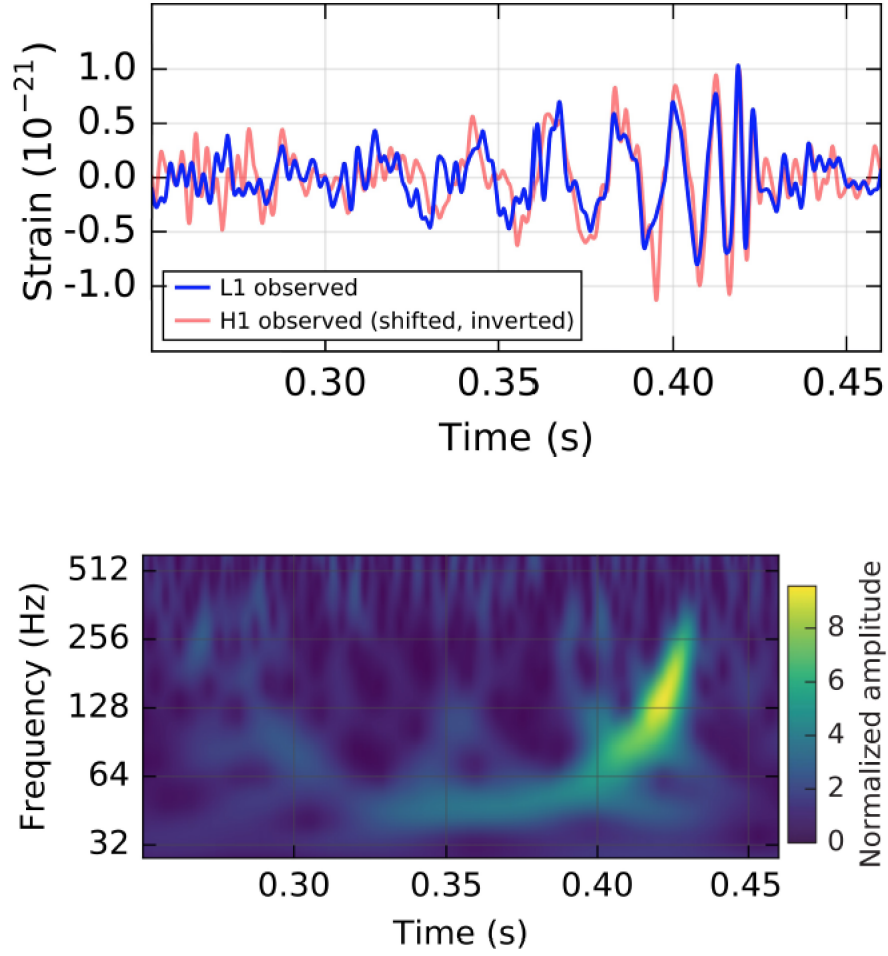


Figure 1: The sensational LIGO discovery of GW150914 published in Abbott et al (2016a). (Top) The results from the two LIGO observatories are overlaid. (Bottom) The L1 data presented as the frequency of the gravitational radiation with time. The strain amplitude is colour coded.

where the dimensionless constant of proportionality ( $32/5$ ) requires some general relativity (metric tensors). The radiated energy must be equal to the change in total energy such that  $P_{\text{rad}} = -dE_{\text{tot}}/dt$  which brings in the time derivative  $\dot{\omega}$ . By substitution, it follows that

$$\frac{(Mm)^{3/5}}{(M+m)^{1/5}} = \frac{c^3}{G} \left( \frac{5}{96} \omega^{-11/3} \dot{\omega} \right)^{3/5}. \quad (6)$$

The LHS of this equation is referred to as the chirp mass  $\mathcal{M}$ : this has the form of a mass as is seen by setting  $m = M$ . The summed masses  $M+m$  are always greater than or equal to  $2\mathcal{M}$ . If we insert the frequency  $f$  ( $= \omega/\pi$ ) of the observed gravitational radiation into the above equation then

$$\mathcal{M} = \frac{c^3}{G} \left( \frac{5}{96} \pi^{-8/3} f^{-11/3} \dot{f} \right)^{3/5} \quad (7)$$

which describes a rapid chirp through the modulated frequency dependence. The increase in amplitude comes from the dependence of  $P_{\text{rad}}$  on the frequency  $f$ . As Mathur et al (2016) observe, this is the precise equation quoted in the LIGO discovery paper (Abbott et al 2016a). As Abbott et al (2016b) show, Eq. 7 can be integrated with respect to time to allow  $f$  to be expressed as an explicit function of time. This is plotted in Fig. 2 along with a simple prediction for the strain amplitude (Abbott et al 2016b).

It is interesting to note the paper by Regge & Wheeler (1957) on the stability of the Schwarzschild singularity. After ring down, it seems likely that the merged black hole continues to oscillate forever but with small perturbations. Maybe one day our descendants will be able to detect the low-amplitude oscillatory behaviour of all black holes. By then, we will have learnt to “image” gravitational waves from the Universe, to measure the cosmic background and to form spherical harmonics in both polarisation states. There is no end to physics.

### 3 Chirp signal from a spinning coin

Spinning coins are a nice demonstration of a finite-time singularity. As van den Engh et al (2000) observe, “the familiar shuddering motions of spinning coins as they come to rest are not at all intuitive.”

Assume a coin with a rounded (bevelled) rim is rolling on its edge over a flat surface without slipping and with an angular velocity  $\omega_r$ . Once the coin slows enough, it starts to tip over and wobble about a vertical axis perpendicular to the flat surface. The coin’s motion is now a combination

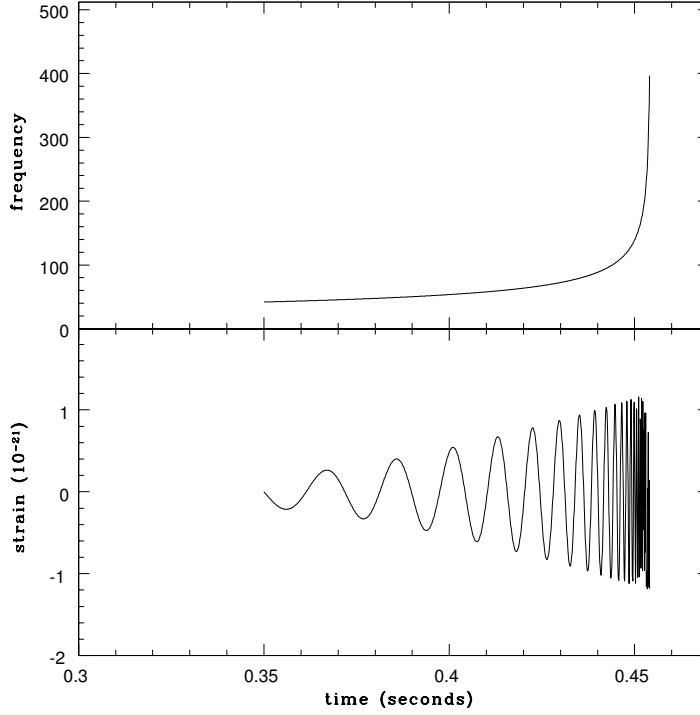


Figure 2: Simple model to compare with the LIGO discovery of the black hole binary merger in Fig. 1. (Top) The solution to Eq. 7 after integration where the initial condition is set by the data at  $t = 0.35$  sec in Fig. 1; (Bottom) The predicted strain amplitude (normalized to the second derivative of the quadrupole moment). The simple formulae use only Newtonian dynamics (with enforced quadrupole emission) and do not accommodate any form of damping (ring down).

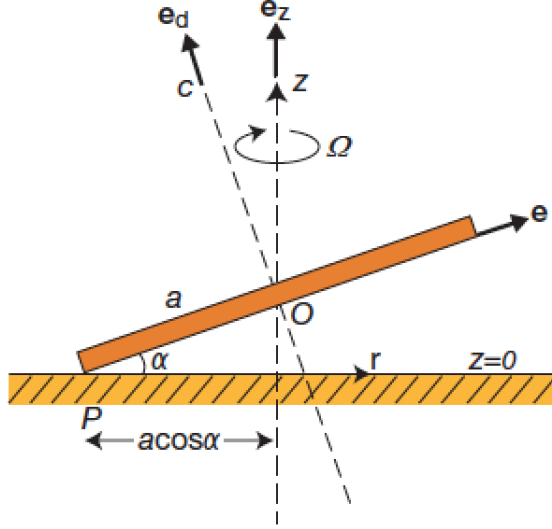


Figure 3: Geometry of a spinning coin (Moffatt 2000):  $\Omega$  is the total angular velocity (rolling+wobble) of the spinning disk about the vertical axis. This is solved in terms of the components about  $\vec{e}_d$  (rolling) and  $\vec{e}$  (wobble).

of rolling ( $\omega_r$ ) and precession defined by an angular velocity ( $\omega_p$ ) about the vertical axis, i.e. the precession of the disk rolling vector about the vertical axis. If the coin's tilt angle is  $\alpha$  with respect to the surface, the circle drawn by the rim of the precessing disk is smaller than the coin's perimeter. Thus for  $\alpha < \pi/2$

$$\omega_r = \omega_p(\sec \alpha - 1) \quad (8)$$

As  $\alpha$  gets smaller,  $\omega_r \rightarrow 0$  and the coin settles into a rapid sequence of wobbles ( $\omega_p$  rising) before it stops abruptly for reasons that are not captured in Eq. 8. Different authors have contrasting views on what causes the coin wobble to halt rather than rise asymptotically to infinite frequency. For example, Moffatt (2000) argues that air viscosity is dominant, although Bildsten (2002) finds that contact forces are more likely to dominate. We discuss these effects below.

Moffatt (2000) has presented a dynamical analysis and we use his notation in Fig. 3. He considers the total angular momentum about a vertical axis as the sum of the spinning disk about its spin axis, and the wobbling disk about an axis in the plane of the disk. (The line  $OP$  defines an instan-

taneous axis of rotation as the disk wobbles about it.) The angular velocity of the disk  $\vec{\omega}$  acting about the vector  $\vec{e}$  is given by

$$\vec{\omega} = \Omega \cos \alpha \vec{e}_d - \Omega \vec{e}_z \quad (9)$$

Thus  $\omega = \vec{\omega} \cdot \vec{e} = -\Omega \sin \alpha$  and  $\omega = \vec{\omega} \cdot \vec{e}_d = 0$  as expected. The total angular momentum of the disk is  $\vec{h} = A\omega\vec{e}$  where  $A = \frac{1}{4}Ma^2$  is the moment of inertia of a disk of mass  $M$  about its diameter  $2a$ . From Euler's equation, the gravitational torque  $\vec{G}$  about the contact point is given by

$$\vec{G} = \dot{\vec{h}} = \vec{\Omega} \times \vec{h} \quad (10)$$

$$= Mga \vec{e}_z \times \vec{e} \quad (11)$$

which leads to

$$\Omega^2 = 4g/a \sin \alpha \quad (12)$$

$$\Omega^2 \approx 4g/a\alpha \text{ for small } \alpha \quad (13)$$

where  $g$  ( $=980 \text{ cm}^2 \text{ s}^{-1}$ ) is the acceleration due to Earth's gravity.

The finite-time singularity is now clear: as  $\alpha$  decreases due to dissipation, the coin wobble frequency increases asymptotically since  $\Omega = 1/\sqrt{\alpha}$  to infinite frequency. But since nature abhors a singularity, this of course never happens, at least not in this Universe.<sup>2</sup>

On energetic grounds, when the coin is held vertically or simply balanced on its edge, it has excess potential energy with respect to the rest position when lying on its side. The impulse we give to spin the coin defines its maximum (potential + kinetic) energy. As the coin tips over, the spin is communicated to the rolling and precessing motion as we have described:

$$E = \text{PE} + \text{KE} \quad (14)$$

$$= Mga \sin \alpha + \frac{1}{2}Mga \sin \alpha \quad (15)$$

$$= \frac{3}{2}Mga \alpha \text{ for small } \alpha \quad (16)$$

Moffatt (2000) presents a case for damping the asymptotic behaviour through air viscosity. First consider the impact of pushing air out of the way over a distance  $a$  in a time given by the wobble rate. The viscous dissipation of energy over the displaced volume below the coin is  $\nu \approx \pi\mu ga^2/\alpha^2$  where

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<sup>2</sup>There are youtube videos of Euler disks asymptoting to wobble frequencies in excess of 100 Hz which underscores what is meant by a finite-time singularity when seen.

$\mu$  is the air's viscosity; note that  $\nu \rightarrow \infty$  as  $\alpha \rightarrow 0$ . Finally, we equate the change in the total energy with the energy lost to viscosity such that

$$\frac{3}{2}Mga \dot{\alpha} = -\pi\mu ga^2/\alpha^2 \quad (17)$$

which integrates to

$$\alpha^3 = 2\pi\mu a(t_0 - t)/M. \quad (18)$$

This has the desired form that  $\alpha$  goes to zero in finite time  $t_0$  but at the cost of the angular velocity  $\Omega \approx (t_0 - t)^{-1/6}$  becoming singular! Moffatt (2000) observes that the size of the vertical acceleration  $|a\ddot{\alpha}|$  cannot exceed  $g$  such that the above formula breaks down at time  $\tau$  before  $t_0$  such that

$$\tau \approx (2a/9g)^{3/5}(2\pi\mu a/M)^{1/5} \quad (19)$$

Moffatt carried out experiments with settling disks and found that the settle times are about right within 20% of this approximation. Thus he believed that air viscosity was dominant in suppressing the singularity. Interestingly, Moffatt was able to achieve  $\Omega \approx 500$  Hz with a commercially available “Euler disk”, i.e. 4 orders of magnitude change in the wobble frequency, demonstrating beautifully the notion of a quasi-singularity in finite time. Bildsten (2002) constructed a more sophisticated model for the air boundary and derived  $\Omega \approx (t_0 - t)^{-2/9}$ , in better agreement with spinning disk observations ( $\Omega \approx 20 - 70$  Hz) by McDonald & McDonald (2000).

Several authors have questioned Moffatt's viscosity argument. van den Engh et al (2000) notes there are 20% differences observed between vacuum and air at early times of spinning upright, but not during the settling phase. They find that different spinning structures (e.g. annular rings, convex lids) have similar settling times thereby downplaying the role of viscosity, turbulence or other kinds of air flow dynamics. They emphasize the importance of surface friction, for example, by spinning an Euler disk on a table top rather than a smooth surface, and finding that it comes to a halt quickly. They also identify a new kind of “rolling friction” when rough-edged coins are spun on a table top; see also Petrie et al (2002) and Bildsten (2002).

## 4 Experiment: How to spin a coin with no hands

van den Engh et al (2002) provide the interesting fact that the Dutch 2.5-guilder coin has magnetic properties allowing it to be spun at a precise frequency on a magnetic stirrer. Being Australians, thence far removed



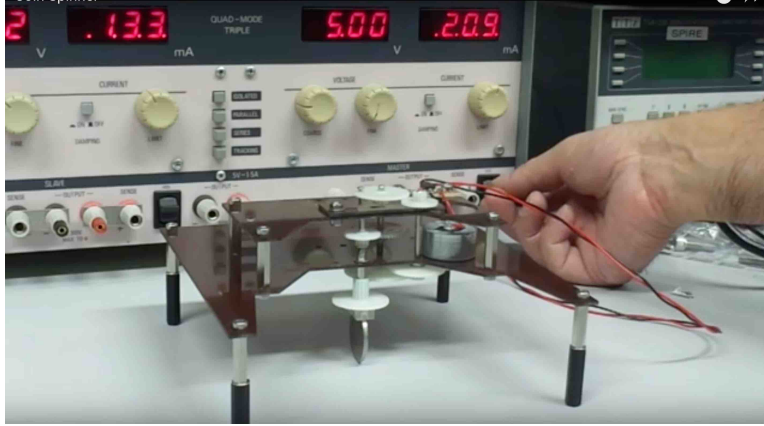


Figure 4: Automated machine to spin a coin used in our experiments below. See the youtube video <https://www.youtube.com/watch?v=uO62dLAcr88> .

from guilders and stirrers, we decided to build a device that could spin an Australian 10 cent coin; see Fig. 4 and Fig. 5 for details of the mechanism. We present a demonstration of the device in a youtube video <sup>3</sup>. It needed to be compact enough to fit into a vacuum chamber at the University of Sydney (15 cm span).

First, we filmed the spinning coin in slow motion in air (240 fps) using an iPhone 6. The sound was also recorded with an iPhone using the Recorder Plus app. See the movie referred to under Footnote 1. These data can be read into Matlab or Python and analysed.

In our experiment, we record the sound of the spinning coin in air.<sup>4</sup> Each time the spin motion is directed towards the microphone, the burst of sound generates a sound packet that we resolve with the iPhone. The far side sound is partly absorbed by the coin. Thus during the coin spinning, we see a succession of sound packets that become more compressed (increasing instantaneous wobble frequency) and the amplitude increases dramatically. In Fig. 6, the rising amplitude is approximately in phase with the rising sine wave. The lower plot shows how the peaks and nulls both line up with the positive peaks of the squared wave. This behaviour is specific to a sound experiment. If we had used a laser pointer illuminating a spinning coin, for example, the reflected light leads to an oscillating pattern at twice

<sup>3</sup><https://www.youtube.com/watch?v=uO62dLAcr88>

<sup>4</sup>We considered experiments based on reflected light but these are more problematic.

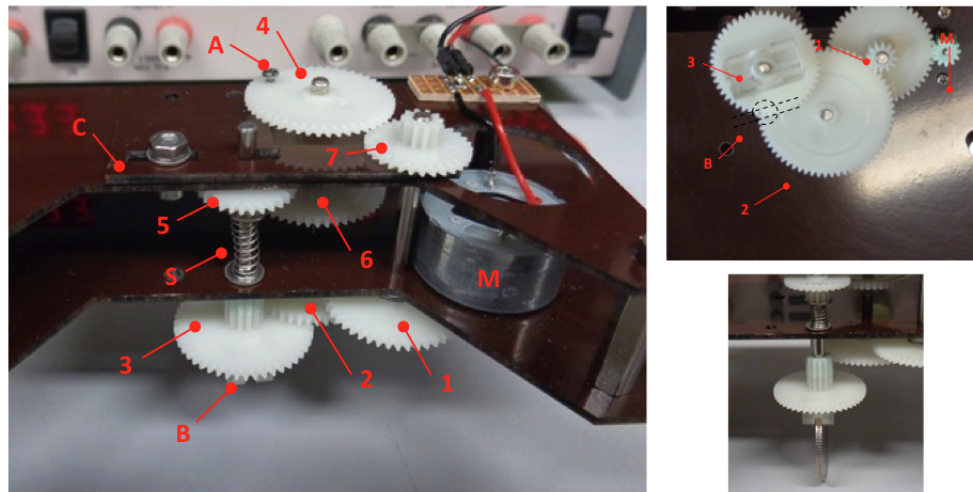


Figure 5: The coin spinner mechanism is operated from a single motor M. The pinion on M turns gear 1 on shaft 1 [12:48 ratio]; gear 1 turns gear 2 on shaft 2 [12:60 ratio]; and gear 2 turns gear 3 on shaft 3 [60:12 ratio]. Gear 3 can be depressed against spring S to engage and hold a coin in block B, and is locked in this position by sliding plate C to the right. As shaft 2 rotates clockwise, pin A on gear 4 pushes plate C back towards the left allowing shaft 3 to spring up, thereby releasing the spinning coin. Wheels 5 and 6 form stops to prevent shaft 3 from being depressed too far. Wheel 7 simply holds shaft 1 in position. The coin holder block B is machined from Perspex: 16x8x6 mm (LWH) with a narrow, 2 mm deep slot running its length and a 5 mm diameter hole through hole; the slot allows easy placement of the coin while the hole helps locate it centrally under the shaft. The mechanism plates are machined from high gloss, low friction MDF drill entry/exit backing boards (e.g. part number 700-015-1 from MegaUK.com).

the frequency and in phase with all, rather than alternating, peaks of the squared wave.

## 5 Vacuum chamber experiment

We were able to obtain slow-motion movies of a coin spinning in a vacuum; the link to the movie is given in Footnote 1. The chamber has vacuum seals that allow for electronic control of a remote device, as shown. The chamber can be evacuated to 1/7000 of an Earth’s atmosphere in about 2 mins. This allows for rapid cycling of the chamber for repeating spin tests. We find that the reduced air resistance allows for the coin to spin upright for about 40% longer under highly repeatable conditions, but the final settle phase is not changed compared to atmospheric conditions. We do not refer to this as the “ring down” phase by analogy with LIGO because the sound amplitude is still increasing.

## 6 Comparing chirps

We have considered two problems in analytic dynamics which exemplify the remarkable phenomenon of chirp. The advantage of the coin experiment is that the data are easily gathered from repeated experiment, and easily analysed. To keep our report to a manageable length, we have refrained from including the power spectrum analysis of either the LIGO or the coin data (e.g. Flandrin 2000). While it is possible for a student to manipulate processed LIGO data (e.g. Fig. 1), the analysis of raw data is far more involved and not broached here. For the spinning coin experiment, a refinement would be to attach vibration sensors (and maybe heat sensors) to the metal base to record all contact. Another improvement is to use a magnetic disk spun by a magnetic stirrer because the torque impulse can be specified accurately. It will be possible to temporarily resolve many forms of contact, slippage, etc. This will keep the student engaged over a month or so. After that, they will be fired up for original research based on more challenging experiments.

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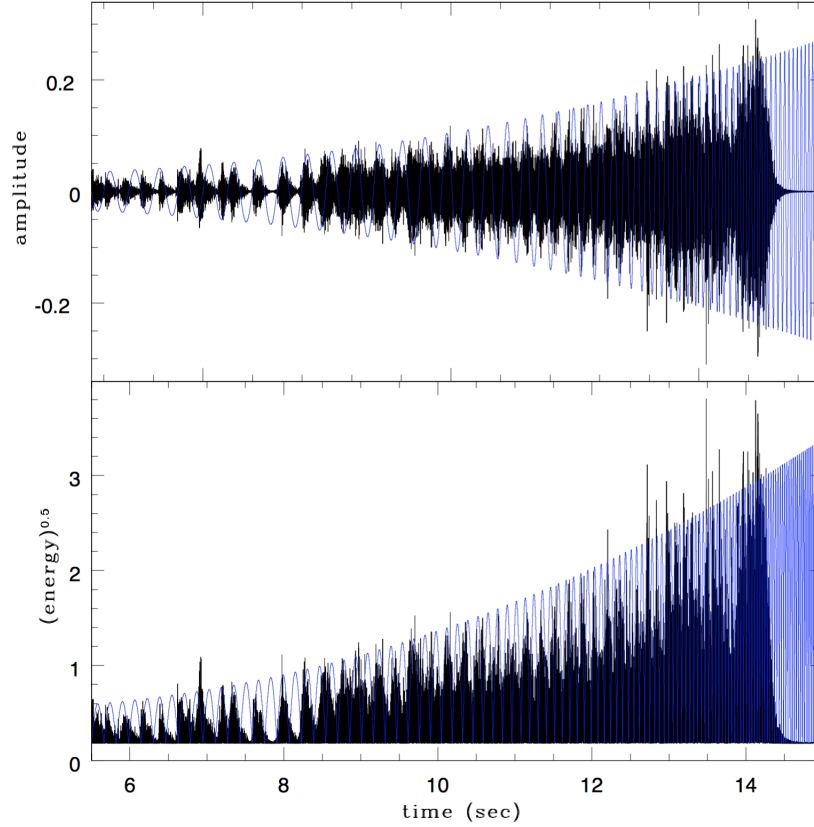


Figure 6: (Top) Sound recording shown as amplitude vs time of a spinning coin. As the contact point spins into view, the sound rises (rising sine) before fading as it recedes (falling sine). The rise and fall are approximately represented by a chirped sine wave - see lower figure. (Bottom) Square root of the squared amplitude to emphasize the energy peaks of the sound packets. The instantaneous frequency spacing of these peaks is shown in Fig. 7. The peaks and nulls are supposed to line up with the sine wave peaks but this is not always the case due to imperfections in our set up.

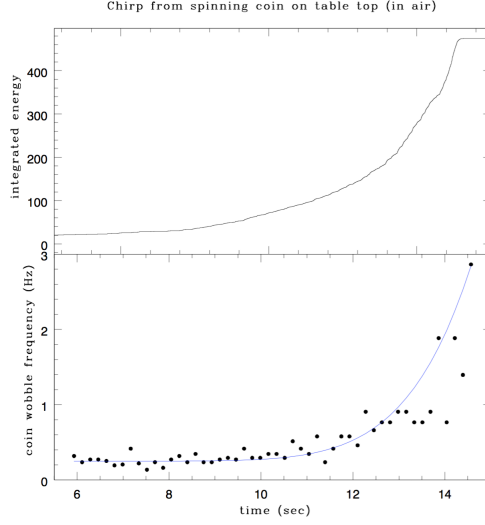


Figure 7: (Top) The integrated energy in Fig. 6 with time. (Bottom) The increase in the instantaneous frequency in Fig. 6 with time.

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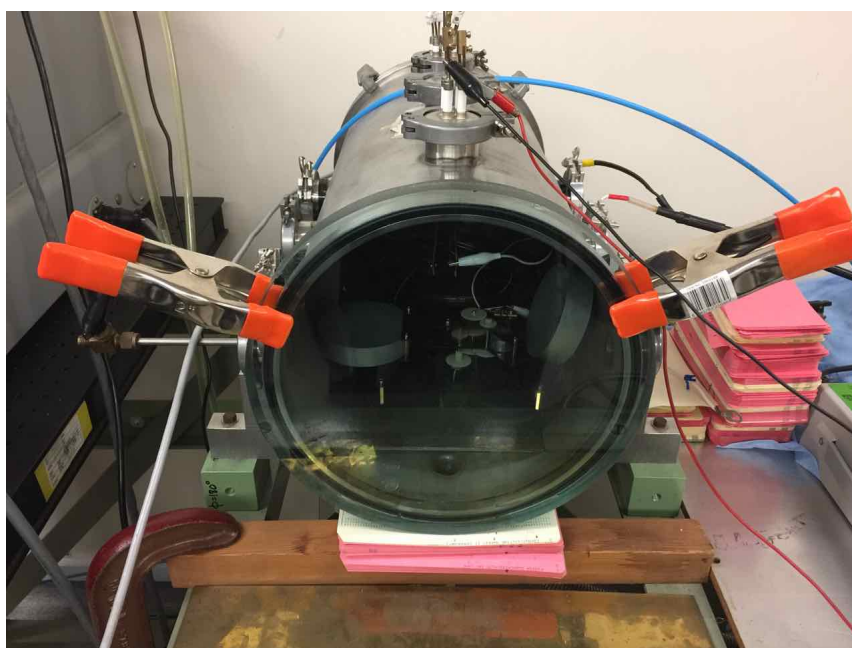


Figure 8: The automated coin spinner loaded into the vacuum chamber in the Space Lab at the School of Physics, University of Sydney.